EXTENSION OF CR MAPS OF POSITIVE CODIMENSION

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Abstract. We study the holomorphic extendability of smooth CR maps between real analytic strictly pseudoconvex hypersurfaces in complex affine spaces of different dimensions.

1. Introduction

This paper concerns the following long-standing conjecture: let $f : M \longrightarrow M'$ be a smooth CR map between two real analytic strictly pseudoconvex hypersurfaces in the complex affine spaces $\mathbb{C}^n$ and $\mathbb{C}^N$ respectively with $1 < n \leq N$. Then f extends holomorphically to a neighborhood of M. At present the strongest result is due to Forstneric [3] who proved that f extends to a neighborhood of an open dense subset of M.

Here we prove the following

Theorem 1.1. *Let $M \subset \mathbb{C}^n$, $M' \subset \mathbb{C}^N$ be C^ω strictly pseudoconvex hypersurfaces and $f : M \longrightarrow M'$ be a C^∞ CR map. If $2 \leq n \leq N < 2n$ then $f \in \mathcal{O}(M)$.*

This gives a complete solution to the above problem in the case where the "codimension" $N - n$ of the map f is smaller than n.

Aknowledgements. This work was initiated when the first author visited the Universities Aix-Marseille-I and Lille-I in May 2004 and completed when the second author visited the University of Illinois during the Spring semester 2005. They thank these institutions.

2. Notations and preliminaries

Denote by $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $z' = (z'_1, ..., z'_N) \in \mathbb{C}^N$ the standard coordinates in $\mathbb{C}^n$ and $\mathbb{C}^N$ respectively. Without loss of generality we may assume that $0 \in M$, $0' \in M'$ and $f(0) = 0'$. It is enough to prove that f extends holomorphically to a neighborhood of the origin.

Consider sufficiently small connected neighborhoods $\mathcal{U}$ and $\mathcal{U}'$ of 0 and $0'$ respectively. Let $\rho(z) \equiv \rho(z, \overline{z}) \in C^\omega(\mathcal{U})$ and $\rho'(z') \equiv \rho'(z', \overline{z}') \in C^\omega(\mathcal{U}')$ be strictly plurisubharmonic defining functions of M and M' respectively. We will denote by $\rho(z, \overline{w})$, $\rho'(z', \overline{w}')$ their complexifications. If $\omega = \overline{w}$, $\omega' = \overline{w}'$, then $\rho(z, \omega) \in \mathcal{O}(\mathcal{U} \times \mathcal{U})$, $\rho'(z', \omega') \in \mathcal{O}(\mathcal{U}' \times \mathcal{U}')$.

For $w \in \mathcal{U}$ denote by $Q_w := \{z \in \mathcal{U} : \rho(z, \overline{w}) = 0\}$ the Segre variety of w. The Segre variety $Q'_{w'}$ is defined similarly for $w' \in \mathcal{U}'$. Consider also the one-sided neighborhoods

$$\mathcal{U}^+ := \{z \in \mathcal{U} : \rho(z) > 0\}, \mathcal{U}^- := \{z \in \mathcal{U} : \rho(z) < 0\},$$
$$\mathcal{U}'^+ := \{z' \in \mathcal{U}' : \rho'(z') > 0\}, \mathcal{U}'^- := \{z' \in \mathcal{U}' : \rho'(z') < 0\}$$

Then f extends holomorphically to $\mathcal{U}^-$, and we may assume that $f(\mathcal{U}^-) \subset \mathcal{U}'^-$, $f \in C^\infty(\mathcal{U}^- \cup M)$. Furthermore, by Forstneric [3] there exists an open dense subset $\Sigma \subset M \cap U$ such that $f \in \mathcal{O}(\mathcal{U}^- \cup \Sigma)$.

Date: 2008-2-2.

2000 *Mathematics Subject Classification.* 32H02, 53C15.

If $a \in \Sigma$ and f is holomorphic on a neighborhood V of a, then $\rho'(f(z)) \in C^\omega(V)$ and $\rho'(f(z)) = \alpha(z)\rho(z)$ for $\alpha(z) \in C^\omega(V)$. After the complexification we have $\rho'(f(z), \overline{f(w)}) = \alpha(z, \overline{w})\rho(z, \overline{w})$. This implies that for w close enough to $a \in \Sigma$ we have

$$f(Q_w \cap V) \subset Q'_{f(w)} \tag{2.1}$$

Thus, if f extends holomorphically across M, then the graph of the extended map f over $\mathcal{U}^+$ must be contained in the set

$$F := \{(w, w') \in \mathcal{U}^+ \times \mathcal{U}' : f(Q_w \cap \mathcal{U}^-) \subset Q'_{w'}\} \tag{2.2}$$

(Notice that since M is strictly pseudoconvex and $\mathcal{U}$ is a small neighborhood of the origin, then $Q_w \cap \mathcal{U}^-$ is connected, see [5, 6]).

The set F has already been used by Forstneric in [3] and our proof of Theorem is based on the result of [3] and further careful study of F.

If $d(z, M)$ denotes the euclidean distance from $z \in \mathcal{U}$ to M, then it is wellknown that if the map f is *non-constant* then for $z \in \mathcal{U}^-$

$$d(f(z), M') \sim d(z, M) \tag{2.3}$$

which here and later means that there exists a constant $c > 0$ such that

$$\frac{1}{c} d(z, M) \leq d(f(z), M') \leq c d(z, M) \tag{2.4}$$

for all $z \in \mathcal{U}^-$. The left part of (2.4) is the consequence of the Hopf lemma while the right part follows from the assumption $f \in C^\infty(M)$. Another wellknown fact is that in this case the differential df has maximal rank near M, i.e. we may assume that $f : \mathcal{U}^- \longrightarrow \mathcal{U}'$ is an embedding.

Consider a C^∞ extension of f to $\mathcal{U}$ which we denote by $\tilde{f}$. We may assume that $\tilde{f} : \mathcal{U} \longrightarrow \mathcal{U}'$ is a proper embedding and thus $S' := \tilde{f}(\mathcal{U}) \subset \mathcal{U}'$ is a closed C^∞ manifold extending $f(\mathcal{U}^-)$.

Lemma 2.1. *Let ρ be a strictly plurisubharmonic C^∞ function on $\mathcal{U}$. Then (after shrinking $\mathcal{U}$)*

$$\rho(z, \overline{z}) + \rho(w, \overline{w}) - \rho(z, \overline{w}) - \rho(w, \overline{z}) \sim |z - w|^2 \tag{2.5}$$

for $z, w \in \mathcal{U}$.

Proof. Let $\rho(z, \overline{z}) = \sum_{k,l} c_{k,l} z^k \overline{z}^l$ be the Taylor expansion of ρ at 0 in the multi-index notation. Since ρ is a real function, we have $c_{lk} = \overline{c}_{kl}$. Let also

$$\phi(z, \overline{w}) := \sum c_{kl}(z^k \overline{z}^l + w^k \overline{w}^l - z^k \overline{w}^l - w^k \overline{z}^l) = \sum c_{kl}(z^k - w^k)(\overline{z}^l - \overline{w}^l)$$

Then

$$\rho(z, \overline{z}) + \rho(w, \overline{w}) - \rho(z, \overline{w}) - \rho(w, \overline{z}) = \phi(z, \overline{w}) = L(z - w) + o(|z - w|^2)$$

where $L(z - w)$ here denotes the Levi form of ρ at 0 which satisfies $L(z - w) \sim |z - w|^2$.

Corollary 2.2. *In the situation of lemma 2.1 there exists a constant $c > 0$ such that for any $z \in \mathcal{U}^-$ we have the inclusion $Q_z \cap \mathcal{U} \subset \mathcal{U}^+$ and*

$$d(Q_z \cap \mathcal{U}, M) \geq c d(z, M) \tag{2.6}$$

Proof. For any $w \in Q_z$ we have $\rho(z, \overline{w}) = \rho(w, \overline{z}) = 0$. By (2.5) we have $\rho(z, \overline{z}) + \rho(w, \overline{w}) \geq 0$. Since $d(z, M) \sim |\rho(z, \overline{z})|$ and $\rho(z, \overline{z}) < 0$ this implies $\rho(w, \overline{w}) > 0$ (i.e. $w \in \mathcal{U}^+$) and $d(w, M) \geq c d(z, M)$.

Corollary 2.3. *If that $z \in M \cap \mathcal{U}$ and $w \in Q_z \cap \mathcal{U}$, then*

$$d(w, M) \sim |w - z|^2$$

This directly follows from (2.5).

Lemma 2.4. *F is an analytic set in $\mathcal{U}^+ \times \mathcal{U}'$ of dimension $\geq n$.*

Proof. We assume that $\frac{\partial \rho}{\partial z_1}(0,0) \neq 0$ and therefore for any $w \in \mathcal{U}$ the equation $\rho(z, \overline{w}) = 0$ of Q_w is equivalent to $z_1 = h(z_2, ..., z_n, \overline{w})$, where h is holomorphic in $z_2, ..., z_n$ and antiholomorphic in w. Thus for $w \in \mathcal{U}^+$ the condition $f(Q_w \cap \mathcal{U}^-) \subset Q'_{w'}$ is equivalent to the condition that for every $z = (z_1, ..., z_n) \in Q_w \cap \mathcal{U}^-$

$$\rho'(f(h(z_2, ..., z_n, \overline{w}), z_2, ..., z_n), \overline{w}') = 0 \tag{2.7}$$

This is a system of (anti)holomorphic equations for w, w'. Since F is obviously closed in $\mathcal{U}^+ \times \mathcal{U}'$, it is an analytic set. If w, z are close to a point of holomorphic extendability of f, then $\rho(z, \overline{w}) = 0$ implies $\rho'(f(z), \overline{f(w)}) = 0$ and thus F contains a piece of the graph of the extension of f and $\dim F \geq n$.

3. Boundary behaviour of F

Let $\overline{F}$ be the closure of F in $\mathcal{U} \times \mathcal{U}'$ and π, π' be the natural projections of $\mathcal{U} \times \mathcal{U}'$ to $\mathcal{U}$ and $\mathcal{U}'$ respectively.

First notice that if $(w, w') \in \overline{F}$ and $w \in M$, then $w' \in Q'_{f(w)}$. Indeed, let $(w, w') = \lim_{\nu \longrightarrow \infty}(w^\nu, w'^\nu)$, $(w^\nu, w'^\nu) \in F$, and $z^\nu \in Q_{w^\nu} \cap \mathcal{U}^-$. Consider a sequence $z^\nu \in Q_{w^\nu} \cap \mathcal{U}^-$ such that $z^\nu \longrightarrow w$ and so $f(z^\nu) \longrightarrow f(w)$. Then $f(z^\nu) \in Q'_{w'^\nu}$ and $w'^\nu \in Q'_{f(z^\nu)} \longrightarrow Q'_{f(w)}$ so that $w' \in Q'_{f(w)}$. This can be reformulated as

$$\overline{F} \cap (\{w\} \times \mathcal{U}') \subset \{w\} \times Q'_{f(w)} \tag{3.1}$$

for $w \in M \cap \mathcal{U}$.

We will now improve (3.1). Differentiating (2.7) with respect to z_k, $k = 2, ..., n$ we get

$$\sum_{j=1}^{N} \rho'_j(f(z), \overline{w}') \left(\frac{\partial f_j}{\partial z_1}(z) \frac{\partial h}{\partial z_k}(\tilde{z}, \overline{w}) + \frac{\partial f_j}{\partial z_k}(z) \right) = 0$$

where $\tilde{z} = (z_2, ..., z_n)$ and $\rho'_j := \frac{\partial \rho'}{\partial z'_j}$. Since

$$\frac{\partial h}{\partial z_k}(\tilde{z}, \overline{w}) = -\frac{\rho_k(z, \overline{w})}{\rho_1(z, \overline{w})}$$

for $z \in Q_w \cap \mathcal{U}$, this is equivalent to

$$\sum_{j=1}^{N} \rho'_j(f(z), \overline{w}') T_k f_j(z, \overline{w}) = 0, k = 2, ..., n \tag{3.2}$$

where

$$T_k f_j(z, \overline{w}) := \rho_1(z, \overline{w}) \frac{\partial f_j}{\partial z_k} - \rho_k(z, \overline{w}) \frac{\partial f_j}{\partial z_1}(z) \tag{3.3}$$

In particular, if $w \in M$ we can take $z = w$ and (3.2) becomes

$$\sum_{j=1}^{N} \rho'_j(f(w),\overline{w}')T_k f_j(w,\overline{w}) = 0, k = 2,...,n$$

Thus we proved

Lemma 3.1. *If $w \in M \cap \mathcal{U}$ then*

$$\overline{F} \cap (\{w\} \times \mathcal{U}') \subset \{w\} \times \{w' \in \mathcal{U}' \cap Q'_{f(w)} : \sum_{j=1}^{N} \rho'_j(f(w),\overline{w}')T_k f_j(w,\overline{w}) = 0, k = 2,...,n\}$$

Consider now a C^∞ extension $\tilde{f}$ of f to $\mathcal{U}$. Since df has maximal rank at 0 we may assume that it remains maximal in $\mathcal{U}$ and $\tilde{f}$ is a proper embedding of $\mathcal{U}$ to $\mathcal{U}'$. The image $S' = \tilde{f}(\mathcal{U}) \subset \mathcal{U}'$ is a C^∞ manifold of real dimension $2n$ which extends $f(\mathcal{U}^-)$.

Lemma 3.2. *For $(w,w') \in \overline{F}$ with $w \in M \cap \mathcal{U}$*

$$d(w',S') \sim |w' - f(w)| \tag{3.4}$$

Proof. Choose the local coordinates near $0 \in \mathbb{C}^n$ and $0' \in \mathbb{C}^N$ such that

$$\rho(z) = 2x_1 + |z|^2 + o(|z|^2), \rho'(z') = 2x'_1 + |z'|^2 + o(|z'|^2) \tag{3.5}$$

$$f_j(z) = z_j + o(|z|), j = 1,...,n, \tag{3.6}$$

$$f_j(z) = o(|z|), j = n+1,...,N \tag{3.7}$$

and denote $'z' = (z'_1,....,z'_n)$, $''z' = (z'_{n+1},...,z'_N)$ so that $z' = ('z','' z')$.

For $w \in M \cap \mathcal{U}$ let

$$\sigma_w = \{w' \in \mathcal{U}' \cap Q'_{f(w)} : \sum_{j=1}^{N} \rho'_j(f(w),\overline{w'})T_k f_j(w,\overline{w}) = 0, k = 2,...,n\}$$

It follows from (3.3),(3.5),(3.6) that for $w \in M \cap \mathcal{U}$ the sets σ_w are complex manifolds of dimension $N-n$ which smoothly depend on w. Moreover, $f(w) \in \sigma_w$ and $T_{0'}\sigma_0 = \{'z' = 0\}$. By (3.6) we have $T_{0'}(S') = \{''z' = 0\}$ and therefore S' and σ_0 intersect transversally at $0'$. Thus $T_{f(w)}S'$ and $T_{f(w)}\sigma_w$ intersect also transversally and $S' \cap \sigma_w = \{f(w)\}$. This implies (3.4).

Remark. Suppose that the coordinates in $\mathbb{C}^N$ are "normal" for M' at $0'$, i.e. the defining function of M' can be chosen in the form

$$\rho'(z',\overline{z}') = 2x'_1 + \sum_{j=2}^{N} |z'_j|^2 + \sum_{|K|,|L| \geq 2} c_{KL}(y'_1)\tilde{z}^K \overline{\tilde{z}^L} \tag{3.8}$$

where $\tilde{z}' = (z'_2,...,z'_N)$. Then $\sigma_0 = \{'z' = 0\}$ and by lemma 3.1

$$\overline{F} \cap (\{0\} \times \mathcal{U}') \subset \{0\} \times \{'z' = 0\} \tag{3.9}$$

Set $\varphi_c(w,w') = \rho(w) + \rho'(w') - c[d(w',S')]^2$ and $\Gamma_c = \{(w,w') \in \mathcal{U} \times \mathcal{U}' : \varphi_c(w,w') = 0\}$. Since $\mathcal{U}$ and $\mathcal{U}$ are small, $\varphi_c \in C^\infty(\mathcal{U} \times \mathcal{U}')$ and Γ_c is a C^∞ hypersurface passing through $(0,0')$.

Lemma 3.3. *For any $c > 0$ the restriction of the Levi form of φ_c to the complex tangent plane $T_{(0,0')}\Gamma_c$ has at least $2n-1$ positive eigenvalues.*

Proof. Since the tangent plane $T_{0'}(S')$ is an n-dimensional complex plane, the Levi form of the function $[d(w', S')]^2$ at $0'$ has n zeros and $N - n$ positive eigenvalues. Thus for any $c > 0$ the Levi form of $\varphi_c(w, w') = \rho(w) + \rho'(w') - c[d(w', S')]^2$ at $(0, 0')$ has at least $n + N - (N - n) = 2n$ positive eigenvalues and its restriction to $T^c_{(0,0')}\Gamma_c$ has $\geq 2n - 1$ positive eigenvalues.

Let $\Omega_c = \{(w, w') \in \mathcal{U} \times \mathcal{U}' : \varphi_c(w, w') > 0\}$.

Lemma 3.4. *Let $\mathcal{U}$ and $\mathcal{U}'$ be small enough neighborhoods of 0 and $0'$ respectively. For $c > 0$ large enough the intersection $F \cap \Omega_c$ is closed in Ω_c.*

Proof. If $(w, w') \in \overline{F}$ with $w \in M \cap \mathcal{U}$ then $w' \in Q'_{f(w)}$ and by corollary 2.3 applied to M' we obtain $\rho'(w') \leq c_1|w' - f(w)|^2$. By lemma 3.2 $|w' - f(w)|^2 \leq c_2[d(w', S')]^2$ and hence $\rho'(w') \leq c_1c_2[d(w', S')]^2$. Thus, if $(w, w') \in \mathcal{U} \times \mathcal{U}'$ is a limit point for $F \cap \Omega_c$ and does not belong to F, then $\rho(w) = 0$, $\rho'(w') \leq c_1c_2[d(w', S')]^2$ and (w, w') does not belong to Ω_c for $c \geq c_1c_2$.

4. Reflection of analytic sets

Let $\mathcal{U}$, $\mathcal{U}'$, ρ, ρ', M, M' be the same as in the previous section and $(a, a') \in \mathcal{U} \times \mathcal{U}'$. We can find an arbitrary small neighborhood $\Omega = \Omega(a, a') \subset \mathcal{U} \times \mathcal{U}'$ of (a, a') and a neighborhood $V \times V' \subset \mathcal{U} \times \mathcal{U}'$ of $Q_a \times Q_{a'}$ such that for any $(w, w') \in V \times V'$ the intersection $(Q_w \times Q'_{w'}) \cap \Omega$ is non-empty and connected.

For such Ω, a neighborhhod $V \times V'$ and a closed set $A \subset \Omega$ we define its *reflection* $r(A)$ by

$$r(A) := \{(w, w') \in V \times V' : S(w) \subset (\mathcal{U} \times Q'_{w'}) \cap \Omega\} \tag{4.1}$$

where $S(w) := (Q_w \times \mathcal{U}') \cap A$.

Notice that $r(A)$ depends not only on A but also on Ω and $V \times V'$. For fixed Ω, V and V', it follows immediately from (4.1) that $\tilde{A} \subset A$ implies $r(A) \subset r(\tilde{A})$.

If $(b, b') \in Q_a \times Q_{a'}$ and $\Omega(b, b')$ is an appropriate neighborhood of (b, b') then we may consider the *second reflection* $r^2(A) := r(A_1)$ of $A_1 := r(A) \cap \Omega(b, b')$.

Lemma 4.1. *$A \subset r^2(A)$ near (a, a').*

Proof. Let $(z, z') \in A$ be close enough to (a, a'). Then by (4.1) it is enough to show that $A_1 \cap (Q_z \times \mathcal{U}') \subset \mathcal{U} \times Q'_{z'}$. Choose any point $(w, w') \in A_1 \cap (Q_z \times \mathcal{U}')$, i.e. $(w, w') \in r(A) \cap \Omega(b, b')$ and $w \in Q_z$. Since $(z, z') \in A$ and $z \in Q_w$ it follows from (4.1) that $z' \in Q'_{w'}$ and hence $w' \in Q'_{z'}$.

In this paper $A \subset \Omega$ will always be an analytic set. In general, its reflection $r(A)$ is not necessarily a (closed) analytic set in $V \times V'$. However analyticity of $r(A)$ can be established under certain additional conditions. In particular, if $\Omega = \omega \times \omega'$ and A is the graph of a holomorphic map $g : \omega \longrightarrow \omega'$, then $r(A) \subset V \times V'$ is an analytic set defined by the condition $g(Q_w \cap \omega) \subset Q'_{w'}$. This and similar cases have been previously discussed in different papers (see, for instance, [5, 6]). The set F introduced in section 2 of this paper is also a reflection of this kind.

Lemma 4.2. *Let (a, a') be a point of an irreducible analytic set A of dimension d in a neighborhood $\Omega = \Omega(a, a')$. Let $(b, b') \in Q_a \times Q_{a'}$ and $\dim S(b) = \dim(A \cap (Q_b \times \mathcal{U}')) = d - 1$. Then there exists a neighborhood $\Omega(b, b')$ of (b, b') such that after a possible shrinking of $\Omega(a, a')$ the set $r(A) \cap \Omega(b, b')$ is analytic in $\Omega(b, b')$.*

Proof. Since A is irreducible, the set $S(b) = A \cap (Q_b \times \mathcal{U}')$ is an analytic set in $\Omega(a,a')$ of pure dimension $d-1$. There exists a linear change of coordinates in $\mathbb{C}^n_z \times \mathbb{C}^N_{z'}$ such that in the new coordinates $(z^1, z^2) \in \mathbb{C}^{d-1} \times \mathbb{C}^{n+N-d+1}$ we have

$$S(b) \cap \{z^1 = a^1\} = \{(a^1, a^2)\}$$

where (a^1, a^2) are the new coordinates of (a, a'). Consider a neighborhood Ω of (a^1, a^2) of the form $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{C}^{d-1} \times \mathbb{C}^{n+N-d+1}$ such that $S(b)$ has no limit points on $\overline{\Omega}_1 \times \partial\Omega_2$. Then there exists a neighborhood $\Omega(b,b') = \omega(b) \times \omega'(b')$ such that $S(w)$ also does not have limit points on $\overline{\Omega}_1 \times \partial\Omega_2$ for any $w \in \omega(b)$ and therefore the projection $\pi : S(w) \longrightarrow \Omega_1$ is an m-sheeted branched holomorphic covering which depends antiholomorphically on w. There exists an open set $\omega_1 \subset \Omega_1$ such that $S(w) \cap (\omega_1 \times \Omega_2)$ is the union of the graphs of m holomorphic mappings

$$z^2 = g^j(z^1, \overline{w}), z' = g'^j(z^1, \overline{w}), z^1 \in \omega_1, j = 1, ..., m$$

These mappings also depend antiholomorphically on $w \in \omega(b)$.

By the uniqueness theorem the inclusion $S(w) \subset (\mathcal{U} \times Q'_{w'}) \cap \Omega$ is equivalent to the condition $S(w) \cap (\omega_1 \times \Omega_2) \subset (\mathcal{U} \times Q'_{w'}) \cap \Omega$ which can be expressed as

$$\rho'(g'^j(z^1, \overline{w}), \overline{w}') = 0 \tag{4.2}$$

for all $z^1 \in \omega_1$ and $j = 1, ..., m$. This is a family of (anti)holomorphic equations for w, w' and thus $r(A)$ is an analytic set in $\Omega(b, b')$.

5. Proof of Theorem

As in section 2 we assume that ρ, ρ' and f satisfy (3.5), (3.6), (3.7). For any $w = (w_1, w_2, ..., w_n) \in \mathcal{U}$ there exists unique ${}^s w = ({}^s w_1, {}^s w_2, ..., {}^s w_n) \in Q_w$ such that $w_j = {}^s w_j$ for $j = 2, ..., n$. Since by [3] f extends holomorphically across a dense open set $\Sigma \subset M$, it is holomorphic on some open set $\mathcal{U}_1^-$ containing $\mathcal{U}^- \cup \Sigma$. There also exists an open set $\mathcal{U}_1^+$ containing $\mathcal{U}^+ \cup \Sigma$ such that ${}^s w \in \mathcal{U}_1^-$ for any $w \in \mathcal{U}_1^+$. Denote by Q_w^c the connected component of $Q_w \cap \mathcal{U}_1^-$ that contains ${}^s w$. We can now modify the definition of F and consider

$$F_1 := \{(w, w') \in \mathcal{U}_1^+ \times \mathcal{U}' : f(Q_w^c) \subset Q'_{w'}\} \tag{5.1}$$

Obviously F_1 coincides with F over $\mathcal{U}^+$. The proof of lemma 2.4 works for F_1 without any changes and thus F_1 is an analytic set in $\mathcal{U}_1^+ \times \mathcal{U}'$. The intersection $\mathcal{U}_1 := \mathcal{U}_1^+ \cap \mathcal{U}_1^-$ is an open neighborhood of Σ and F_1 contains the graph of f over $\mathcal{U}_1$.

The set F_1 consists of irreducible components of two types. We say that a component of F_1 is *relevent* if it contains an open piece of the graph of f over $\mathcal{U}_1$. Otherwise we call it *irrelevent*. Thus F_1 is the union of two analytic sets: F_r and F_i which consist of all relevent and irrelevent components respectively. It is obvious that the dimension of F_r is $\geq n$ at any its point, the dimension of the intersection of F_i with the graph of f is $< n$ and $(0, 0') \in \overline{F}_r$.

We now represent F_r as $F_r^{(n)} \cup F_r^{(n+1)}$ where $F_r^{(n)}$ is the union of all n-dimensional relevent components and $F_r^{(n+1)}$ consists of all relevent components of dimension $\geq n+1$.

There are two possibilities:

(1) After shrinking $\mathcal{U}$ and $\mathcal{U}'$ we have $F_r = F_r^{(n)}$.
(2) $(0, 0') \in \overline{F_r^{(n+1)}}$.

We first prove Theorem in the second case.

5.1. **Proof of Theorem in the case (2).** We need the following technical statement which is a slight variation of the standard results (see, for instance, [2], p. 36).

Lemma 5.1. *Let A be a complex purely m-dimensional analytic set in a domain $\Omega \subset \mathbb{C}^n$ and (A_ν) be a sequence of purely p-dimensional complex analytic sets in Ω. Suppose that $p \geq m$ and that the cluster set $cl(A_\nu)$ of the sequence (A_ν) is contained in A. Then $p = m$ and $cl(A_\nu)$ is a union of some irreducible components of A.*

As usual, by the cluster set $cl(A_\nu)$ of a sequence (A_ν) we mean the set of all points $a \in \Omega$ such that there exists a subsequence $(\nu(k))$ of indices and points $a_{\nu(k)} \in A_{\nu(k)}$ converging to a as k tends to infinity. For the convenience of readers we give the proof of the lemma.

Proof. Fix a point $a \in cl(A_\nu)$; we can assume that $a = 0$. Consider a complex linear $(n-m)$-dimensional subspace L of $\mathbb{C}^n$ satisfying $A \cap L = \{0\}$. Then there exist a ball B centered at the origin and $r > 0$ such that the distance form A to $L \cap \partial B$ is equal to r. Since $cl(A_\nu) \subset A$, for every ν big enough the sets A_ν do not intersect the $r/2$-neighborhood of $L \cap \partial B$. On the other hand, $0 = \lim a_{\nu(k)}$ with $a_{\nu(k)} \in A_{\nu(k)}$ so for any k big enough the intersection $A_{\nu(k)} \cap B$ is not empty. Then the intersection $(L + a_{\nu(k)}) \cap A_{\nu(k)} \cap B$ is a compact analytic subset in B and so its dimension is equal to 0. Since $\dim L = n - m$, this implies $p = \dim A_{\nu(k)} \leq m$ and we obtain that $m = p$.

Now we prove that $cl(A_\nu)$ coincides with a union of some irreducible components of A. Since the set $\mathcal{S}(A)$ of singular points of A is an analytic set of dimension $< m$, it follows from the first part of lemma that $cl(A_\nu)$ is not contained in $\mathcal{S}(A)$. So the intersection of $cl(A_\nu)$ with the set $\mathcal{R}(A)$ of regular points of A is not empty and this is sufficient to show that $cl(A_\nu)$ is open in $\mathcal{R}(A)$. Consider an arbitrary point $a \in cl(A_\nu) \cap \mathcal{R}(A)$. As above, we assume that $a = 0$. After a biholomorphic change of coordinates we can assume that in a neighborhood of the origin A coincides with the coordinate space P of variables $z_1, ..., z_p$. Denote by L the coordinates space of variables $z_{p+1}, ..., z_n$ and fix small enough the balls $B \subset P$ and $B' \subset L$ centered at the origin. Since $cl(A_\nu) \subset A$, for every ν big enough the set A_ν does not intersect $B \times \partial B'$. So every $A_\nu \cap (B \times B')$ is a analytic covering brunched over B. Hence for every point $b \in B$ the fiber $\{b\} \times L$ contains a point $(b, c_\nu) \in A_\nu \cap (B \times B')$. Since $cl(A_\nu) \subset A$, we get $\lim c_\nu = 0$ which proves the claim.

Lemma 5.2. *If $(0,0') \in \overline{F_r^{(n+1)}}$, then $F_r^{(n+1)}$ extends to an analytic set in a neighborhood of $(0,0')$.*

Proof. Since $F_r^{(n+1)}$ contains only the relevant components, there exists a sequence $w^\nu \in \Sigma$ converging to 0 as $\nu \longrightarrow \infty$ such that $(w^\nu, f(w^\nu)) \in F_r^{(n+1)}$ for any ν (if not, the proof is reduced to the case (1)). Let $\tilde{f}$ be a C^∞ extension of f to $\mathcal{U}$ that coincides with f on $\mathcal{U}_1^-$ and $S' = \tilde{f}(\mathcal{U}) \subset \mathcal{U}'$. Let φ_c, Γ_c and Ω_c be the same as in section 2. Since $d(w', S') = 0$ for $w' = f(w)$, $w \in \mathcal{U}_1^- \cap \mathcal{U}^+$ the intersection $F_r^{(n+1)} \cap \Omega_c$ is not empty and moreover $(0,0') \in \overline{F_c^{(n+1)} \cap \Omega}$. The set $F_r^{(n+1)}$ can be decomposed to a finite union of analytic sets (perhaps, reducible) of pure dimensions $\geq n+1$:

$$F_r^{(n+1)} = \cup_{k \geq n+1} F_{r,k}^{(n+1)}, \dim F_{r,k}^{(n+1)} = k$$

(see, for instance, [2], p.51). By lemma 3.3 the Levi form of φ_c has at least $2n-1$ positive eigenvalues on $T^c_{(0,0')}\Gamma_c$. Since $N < 2n$, the set $F_r^{(n+1)}$ extends to an analytic set in a neighborhood of $(0,0')$ by Rothstein's theorem on the analytic extension across pseudoconcave hypersurfaces (see, for instance, [2]). More precisely, there exists an analytic set $\tilde{F} \subset \mathcal{U} \times \mathcal{U}'$ such that $F_r^{(n+1)} \subset \tilde{F} \cap (\mathcal{U}_1^+ \times \mathcal{U}')$. This set contains the graph of f near $(0,0')$. Every irreducible component of $\tilde{F}$ is of the dimension $\geq n+1$ and has a non-empty open subset contained in $F_r^{(n+1)}$.

Proposition 5.3. *If $(0,0') \in \overline{F_r^{(n+1)}}$, then f extends holomorphically to a neighborhood of* 0.

We begin the proof with the following

Lemma 5.4. *In any neighborhood of $(0,0')$ there exists a point $(w^0, {w'}^0) \in F_r^{(n+1)} \cap (Q_0 \times Q'_{0'})$ with $w^0 \neq 0$, ${w'}^0 \neq 0$. Moreover $(0,0')$ belongs to the closure of some component of $F_r^{(n+1)} \cap (Q_0 \times Q'_{0'})$ which contains $(w^0, {w'}^0)$.*

Proof. Suppose that the coordinates $z' = ('z', ''z')$ in $\mathbb{C}^N$ are "normal" for M' at $0'$, i.e. satisfy (3.8). Consider a sequence $(a^\nu, {a'}^\nu) \in F_r^{(n+1)}$ such that $(a^\nu, {a'}^\nu) \longrightarrow (0,0')$ and for every ν we have $a^\nu \in \mathcal{U}_1^- \cap \mathcal{U}^+$, ${a'}^\nu = f(a^\nu)$. Passing to a subsequence, we may also assume that there exists an irreducible component of $\tilde{F}$ of dimension $d \geq n+1$ containing the graph of f in a neighborhood of $(0,0')$ such that $(a^\nu, {a'}^\nu)$ belongs to this component for every ν. We denote it again by $\tilde{F}$. Let $b^\nu = {}^s a^\nu$. For any $\nu = 1,2,...$ the intersection $\tilde{S}_\nu := \tilde{F} \cap (Q_{b^\nu} \times \mathcal{U}')$ is an analytic set in $\mathcal{U} \times \mathcal{U}'$ of pure dimension $d-1$ and containing $(a^\nu, {a'}^\nu)$. Indeed, if not, $\tilde{F}$ is contained in the hypersurface $Q_{b^\nu} \times \mathcal{U}'$ which is impossible since $\tilde{F}$ contains an open piece of the graph of f. For the same reason the dimension of the set $\tilde{F} \cap (Q_0 \times \mathcal{U}')$ also is equal to $d-1$. The cluster set $\tilde{S}_0 := cl(\tilde{S}_\nu)$ of the sequence $\tilde{S}_\nu$ with respect to $\mathcal{U} \times \mathcal{U}'$ is contained in $\tilde{F} \cap (Q_0 \times \mathcal{U}')$ and so by lemma 5.1 $\tilde{S}_0$ is the union of some components of $\tilde{F} \cap (Q_0 \times \mathcal{U}')$; in particular, $\tilde{S}_0$ is an analytic set of pure dimension $d-1$. On the other hand, denote by F_r^d the union of purely d-dimensional components of $F_r^{(n+1)}$ having a non-empty open intersection with $\tilde{F}$. For any $\nu = 1,2,...$ the intersection $S_\nu := F_r^d \cap (Q_{b^\nu} \times \mathcal{U}')$ is an analytic set in $\mathcal{U}^+ \times \mathcal{U}'$ of pure dimension $d-1$ containing the point $(a^\nu, {a'}^\nu)$ and contained in $\tilde{S}_\nu$. Since $Q_{b^\nu} \subset \mathcal{U}_1^+$ and $F_r^{(n+1)}$ is an analytic set in $\mathcal{U}_1^+ \times \mathcal{U}'$, every S_ν is a (closed) analytic subset in $\mathcal{U} \times \mathcal{U}'$ and so coincides with a union of some irreducible components of $\tilde{S}_\nu$. Therefore by lemma 5.1 the cluster set $S_0 := cl(S_\nu) \subset \overline{F_r^{(n+1)}} \cap (Q_0 \times \mathcal{U}')$ is the union of some components of $\tilde{S}_0$ and $\dim S_0 = d - 1 \geq n$.

On the other hand by the remark after lemma 3.2 $\overline{F_r^{(n+1)}} \cap (\{0\} \times \mathcal{U}') \subset \{0\} \times \sigma_0$, where $\sigma_0 = \{z' \in \mathbb{C}^N : 'z' = 0\}$. Since $\dim S_0 = d - 1 \geq n$ and $\dim \sigma_0 = N - n < n$, the set S_0 is not contained in $\{0\} \times \sigma_0$. Therefore, S_0 is not contained in $\overline{F_r^{(n+1)}} \cap (\{0\} \times \mathcal{U}')$. Hence in any neighborhood of $(0,0')$ there exists a point $(w^0, {w'}^0) \in S_0$ with $w^0 \neq 0$. Moreover, the set S_0 is not contained in $Q_0 \times \{0'\}$ because $\dim S_0 > n - 1 = \dim Q_0$. Therefore, in any neighborhood of $(0,0')$ there exists a point $(w^0, {w'}^0) \in S_0$ with $w^0 \neq 0$, ${w'}^0 \neq 0$. Moreover $w^0 \in \mathcal{U}^+$ because $Q_0 \subset \mathcal{U}^+ \cup \{0\}$. This means that $(w^0, {w'}^0) \in \mathcal{U}^+ \times \mathcal{U}'$ and thus $(w^0, {w'}^0) \in F_r^{(n+1)} \cap S_0$. Finally, it follows from the definition (5.1) that every S_ν is contained in $\mathcal{U} \times Q'_{f(b^\nu)}$. Hence, $S_0 \subset Q_0 \times Q'_{0'}$ and we get the first claim of lemma.

Prove the second claim. After possible shrinking of $\mathcal{U}$ and $\mathcal{U}'$ the set $\tilde{F} \cap (Q_0 \times Q'_{0'}) \times (\mathcal{U} \times \mathcal{U}')$ consists of a finite number of ireducible components and every such component contains $(0,0')$. So the second claim follows from the first part of lemma.

Proof of proposition 5.3: Consider a sequence of points $(w^\nu, {w'}^\nu) \in F_r^{(n+1)} \cap (Q_0 \times Q'_{0'})$ such that $(w^\nu, {w'}^\nu) \longrightarrow (0,0')$ and $w^\nu \neq 0$, ${w'}^\nu \neq 0$. Choose appropriate neighborhoods Ω_ν and Ω_ν^0 of $(w^\nu, {w'}^\nu)$ and $(0,0')$ respectively such that $F_\nu^2 := r(F_r^{(n+1)} \cap \Omega_\nu)$ is an analytic set in Ω_ν^0. Consider the analytic sets $A_\mu := \cap_{\nu=1}^{\mu} F_\nu^2$. Then $A_{\mu+1} \subset A_\mu$. If d_μ denotes the dimension of A_μ, then $d_{\mu+1} \leq d_\mu$. Therefore $d_\mu \geq n$ for every μ and there exists $d_{\mu_0} \geq n$ such that $d_\mu = d_{\mu_0}$ for any $\mu > \mu_0$. Since F_1 defined by (5.1) is the reflection of Γ_f every set F_ν^2 contains an open piece of the graph Γ_f by lemma 4.1. Since the set A_{μ_0} has a finite number of components, there exists a

neighborhood Ω^0 of $(0,0')$ and an irreducible analytic set $F^2 \subset \Omega^0$ containing an open piece of Γ_f such that $\dim F^2 = d_{\mu_0} \geq n$ and for any ν big enough $F^2 \cap \Omega^0_\nu \subset F^2_\nu$. Since F_2 is an analytic set in a neighborhood of $(0,0')$, we can consider its reflection $F_3 := r(F_2)$ which is an analytic set in a neighborhood $\tilde{\Omega}^0$ of $(0,0')$. For any ν big enough $(w^\nu, w'^\nu) \in \tilde{\Omega}^0$. By lemma 4.1 F_3 contains $F_r^{(n+1)}$ near (w^ν, w'^ν). Recall that $F_r^{(n+1)}$ has a finite number of irreducible components in a neighborhood of the origin and every component contains an open piece of Γ_f. Hence, F_3 contains an open piece of Γ_f in a neighborhood of the origin. Thus both F_2 and F_3 are analytic sets in a neighborhood of $(0,0')$ and both contain a piece of Γ_f as well as $F_2 \cap F_3$.

Lemma 5.5. *We have* $F_2 \cap F_3 \cap (\{0\} \times \mathcal{U}') = \{(0,0')\}$.

Indeed, suppose that $\gamma := F_2 \cap F_3 \cap (\{0\} \times \mathcal{U}')$ is an analytic set of positive dimension. Let $(0, z'^0) \in F_2 \cap F_3$. We have $F_3 = \{(z,z') : F_2 \cap (Q_z \times \mathcal{U}') \subset \mathcal{U} \times Q'_{z'}\}$. Hence $F_2 \cap (Q_0 \times \mathcal{U}') \subset \mathcal{U} \times Q'_{z'^0}$ and thus $z'^0 \in Q_{z'^0}$ that is $z'^0 \in M'$. Hence $\gamma \subset M'$ which contradicts the strict pseudoconvexity of M'. This proves the claim.

Therefore $F_2 \cap F_3$ has a locally proper at $(0,0')$ projection $\pi : F_2 \cap F_3 \longrightarrow \mathcal{U}$. Hence $\dim(F_2 \cap F_3) = n$ and $F_2 \cap F_3$ is an analytic continuation of Γ_f to a neighborhood of $(0,0')$. By [1] f extends holomorphically to a neighborhood of $(0,0')$. This completes the proof of proposition 5.3 and proves theorem in the case (2).

5.2. **Proof of Theorem in the case (1).** Consider now the case (1) where $\dim F_r = n$ is a neighborhood of the origin. Everywhere below we suppose that this assumption holds.

Lemma 5.6. *Suppose that*

$$\overline{F}_r^{(n)} \cap (Q_0 \times Q'_{0'}) = \overline{F}_r^{(n)} \cap (\{0\} \times Q'_{0'}).$$

Then

$$\overline{F}_1 \cap (\{0\} \times Q'_{0'}) = \{0\} \times \sigma_0.$$

Proof. Let $\overline{F}_r^{(n)} \cap (\{0\} \times Q'_{0'}) = \{0\} \times X$. It follows by lemma 3.1 that $X \subset \sigma_0$. Consider a sequence (w^ν) of points in Σ converging to 0 and set $w'^\nu = f(w^\nu) \in M'$. Denote by ${}_{w^\nu}Q_z$ the germ of the Segre variety Q_z at w^ν and consider the analytic sets

$$S_\nu = \{(z,z') \in Q_{w^\nu} \times Q_{w'^\nu} : f({}_{w_\nu}Q_z) \subset Q'_{z'}\}$$

in $\mathcal{U} \times \mathcal{U}'$. Then $\dim S_\nu \geq n-1$. Since $\dim \sigma_0 = N - n \leq 2n-1-n = n-1$ and $cl(S_\nu) \subset \{0\} \times X$ (by the hypothesis of lemma) , lemma 5.1 implies that

$$cl(S_\nu) = \{0\} \times X = \{0\} \times \sigma_0$$

which proves lemma.

We claim that the projection of $\overline{F}_r^{(n)} \cap (Q_0 \times Q'_{0'})$ to $\mathbb{C}^n$ can not be equal to the singleton $\{0\}$. Indeed, assume that $\overline{F}_r^{(n)} \cap (Q_0 \times Q'_{0'}) = \{0\} \times \sigma_0$. Fix a point $z'^0 \in \sigma_0$ which does not belong to M' (since M' is strictly pseudoconvex, it contains no analytic sets of positive dimension). Consider a sequence of points $(z^\nu, z'^\nu) \in S_\nu$ converging to $(0, z'^0)$. Consider analytic sets $A_\nu = F_r^{(n)} \cap (Q_{z^\nu} \times Q_{z'^\nu})$. Since $F_r^{(n)}$ contains the graph of f over Σ, for every ν we have $\dim A_\nu \geq n-1$. We have $cl(A_\nu) \subset \{0\} \times \sigma_0$. Hence lemma 5.1 implies $cl(A_\nu) = \{0\} \times \sigma_0$. On the other hand, $cl(A_\nu) \subset \{0\} \times Q'_{z'^0}$. Hence $z'^0 \in \sigma_0 \subset Q'_{z'^0}$ and so $z'^0 \in M'$: a contradiction.

Thus, in any neighborhood of $(0,0')$ the intersection $\overline{F}_r^{(n)} \cap (Q_0 \times Q'_{0'})$ contains points (z,z') with $z \neq 0$. Let us show that for every such point we also have $z' \neq 0$. Indeed, assume by

contradiction that $(z,0) \in \overline{F}_r^{(n)} \cap (Q_0 \times Q'_{0'})$ and $z \neq 0$. Therefore $z \in \mathcal{U}^+$ and $(z,0) \in F_r^{(n)}$. Then $f(Q_z \cap \mathcal{U}^-) \subset Q'_{0'} \subset \mathcal{U}'^+ \cup \{0'\}$. On the other hand, $f(Q_z \cap \mathcal{U}^-) \subset \mathcal{U}'^-$ and $Q'_{0'} \cap \mathcal{U}'^- = \{0\}$. This implies that f vanishes identically on the complex hypersurface $Q_z \in \mathcal{U}^-$ (we point out that Q_z intersects M transversally at the origin since $z \in Q_0$ and $z \neq 0$). However, f has the maximal rank: a contradiction.

We sum up this considerations in the following statement.

Lemma 5.7. *In any neighborhood of* $(0,0')$ *there exists a point* $(w^0, w'^0) \in \overline{F}_r^{(n)} \cap (Q_0 \times Q'_{0'})$ *with* $w^0 \neq 0$ *and* $w'^0 \neq 0$.

Now we are able to conclude the proof of Theorem. Fix a point $(w^0, w'^0) \in F_r^{(n)} \cap (Q_0 \times Q'_{0'})$ with $w^0 \neq 0$ and $w'^0 \neq 0$, fix a neighborhood Ω_0 of (w^0, w'^0) and consider the reflection $F_2 = r(F_r^{(n)} \cap \Omega_0)$. Then F_2 is an analytic set in a neighborhood Ω_1 of $(0,0')$ and contains an open piece of Γ_f; in particular, $\dim F_2 \geq n$. If $\dim F_2 = n$, we conclude. If not, we apply an argument similar to the proof of the case (2).

Consider a basis $\mathcal{U}_\nu \times \mathcal{U}'_\nu$ of neighborhoods of $(0,0')$ and a sequence $(w^\nu, w'^\nu) \in \overline{F}_r^{(n)} \cap (Q_0 \times Q'_{0'}) \cap (\mathcal{U}_\nu \times \mathcal{U}'_\nu)$ with $w^\nu \neq 0$ and $w'^\nu \neq 0'$ such that for every ν there exists a component of $F_r^{(n)} \cap (\mathcal{U}_\nu \times \mathcal{U}'_\nu)$ containing the point (w^ν, w'^ν) and an open subset of Γ_f in $\mathcal{U}_\nu \times \mathcal{U}'_\nu$. Choose appropriate neighborhoods Ω_ν and Ω^0_ν of (w^ν, w'^ν) and $(0,0')$ respectively such that $F^2_\nu := r(F_r^{(n)} \cap \Omega_\nu)$ is an analytic set in Ω^0_ν. As in the proof of the case (2), consider the analytic sets $A_\mu := \cap_{\nu=1}^{\mu} F^2_\nu$. As above let d_μ denotes the dimension of A_μ. Then $d_{\mu+1} \leq d_\mu$ and $d_\mu \geq n$ for every μ; furthermore, there exists $d_{\mu_0} \geq n$ such that $d_\mu = d_{\mu_0}$ for any $\mu > \mu_0$. Since the set F_1 defined by (5.1) is the reflection of graph of f every set F^2_ν contains an open piece of Γ_f in view of lemma 4.1. The set A_{μ_0} has a finite number of components, so there exists a neighborhood Ω^0 of $(0,0')$ and an irreducible analytic set $F^2 \subset \Omega^0$ containing an open piece of Γ_f such that $\dim F^2 = d_{\mu_0} \geq n$ and for any ν big enough $F^2 \cap \Omega^0_\nu \subset F^2_\nu$. Since F_2 is an analytic set in a neighborhood of $(0,0')$, we can consider its reflection $F_3 := r(F_2)$ which is an analytic set in a neighborhood $\tilde{\Omega}^0$ of $(0,0')$. For any ν big enough $(w^\nu, w'^\nu) \in \mathcal{U}_\nu \times \mathcal{U}'_\nu \subset \tilde{\Omega}^0$. By lemma 4.1 F_3 contains $F_r^{(n)}$ near (w^ν, w'^ν). Hence F_3 contains any component of $F_r^{(n)} \cap (\mathcal{U}_\nu \times \mathcal{U}'_\nu)$ passing through (w^ν, w'^ν). Therefore, F_3 contains an open piece of Γ_f. We obtain that both F_2 and F_3 are analytic sets in a neighborhood of $(0,0')$ and both contain an open piece of Γ_f as well as $F_2 \cap F_3$. Repeating the proof of lemma 5.5 we get that the projection $\pi : \tilde{F}_2 \cap F_3 \longrightarrow \mathcal{U}$ is locally proper at $(0,0')$ and we conclude as in the case (2).

References

[1] E.Bedford, S.Bell, *Extension of proper holomorphic mappings past the boundary*, Manuscripta Math. **50**(1985), 1-10.

[2] E.Chirka, *Complex analytic sets*, Kluwer, 1989.

[3] F.Forstneric, *Extending proper holomorphic mappings of positive codimension*, Invent. Math. **95** (1989), 31-61.

[4] K.Diedrich, J.E.Fornaess, *Proper holomorphic mappings between real-analytic pseudoconvex domains in* $\mathbb{C}^n$, Math. Ann. **282**(1988), 681-700.

[5] K.Diederich, S.Pinchuk, *Regularity of continuous CR maps in arbitrary dimension*, Mich. Math. J. **51**(2003), 111-140.

[6] K.Diederich, S.Pinchuk, *Analytic sets extending the graphs of holomorphic mappings*, J. Geom. Anal. **14**(2004), 231-239.

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